Spectral Theory of Eisenstein Series

§1. Motivation for spectral decomposition

__Problem__ Given a Hilbert space H, find a "nice basis" $(e_j)_j$ of H.

$$H \ni v = \sum \langle v, e_j \rangle e_j$$

decomposition of v

$$\langle v_1, v_2 \rangle = \sum \langle v_1, e_j \rangle \langle e_j, v_2 \rangle$$

e.g., $H = L^2(X, \mu)$

(X, μ): probability space

__Example__ $X = \mathbb{R}/\mathbb{Z} \cong [0,1]/\sim$, $0 \sim 1$

circle group

"nice ~~basis~~": $(e_n)_{n \in \mathbb{Z}}$

$$e_n(x) = e^{2\pi i n x} = e(nx), \quad e(x) := e^{2\pi i x}$$

"basis": Theory of Fourier series: e_n give a Hilbert space basis of $H = L^2(X)$ ($\mu =$ Lebesgue measure)

$$L^2(X) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} e_n$$

"Hilbert direct sum": summands orthogonal, span is dense in H

"nice":

- each e_n is an eigenfunction for $\Delta = \frac{\partial^2}{\partial x^2}$

$$\Delta e_n = -(2\pi n)^2 e_n$$

- For $x \in \mathbb{R}/\mathbb{Z}$, let $\rho(x) : H \longrightarrow H$ denote the "translation by x" map: $\forall\, v \in H$,

$$\rho(x) v(y) := v(y + x)$$

Then each e_n is an eigenfunction for each $\rho(x)$:

$$\rho(x) e_n \overset{\checkmark}{=} e(nx) e_n$$

$$: y \mapsto e_n(y+x) \qquad\qquad : y \mapsto e(nx) e_n(y)$$

$$\| \qquad\qquad\qquad \|$$

$$e(n(y+x)) \qquad\qquad e(nx)e(ny)$$

$$\|$$

$$e(nx)e(ny)$$

Said another way, each e_n spans a one-dimensional (in particular, irreducible) subrepresentation $\mathbb{C}e_n \subseteq H$ for the regular representation $\rho : \mathbb{R}/\mathbb{Z} \to GL(L^2(\mathbb{R}/\mathbb{Z}))$.

"e_n transform as simply as possible under translation"

Why care? such bases provide a natural tool for equidistribution problems

Definition Let (X, μ) : Borel probability space.
Let $(x_j)_{j \geq 1}$ be a sequence in X.
We say (x_j) equidistributes (with respect to μ) if $\forall \varphi \in C_c(X)$,

$$(\star) \quad \frac{1}{J} \sum_{1}^{J} \varphi(x_j) \longrightarrow \int_X \varphi \, d\mu \quad \text{as } J \to \infty.$$

$\left(\Longrightarrow \{x_j\}_{j \geq 1} \text{ is dense in } \mathrm{supp}(\mu) \right)$

Weyl's criterion Suppose $(X, \mu) = (\mathbb{R}/\mathbb{Z}, \text{Lebesgue})$.
The following are equivalent $\forall$ (x_j) in X:
(i) (x_j) equidistributes
(ii) $\forall n \in \mathbb{Z}$, $(\star)$ holds for $\varphi = e_n$:

$$(\star\star) \quad \frac{1}{J} \sum_{1}^{J} e(n x_j) \longrightarrow \int_{\mathbb{R}/\mathbb{Z}} e(nx)\,dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

(It suffices to check this when $n \geq 1$.)

Proof $(i) \Rightarrow (ii)$: immediate: take $\psi = e_n$

$(ii) \Rightarrow (i)$: let $\psi \in C_c(\mathbb{R}/\mathbb{Z})$. By the theory of Fourier series, $\forall\, \varepsilon > 0$ $\exists\, \psi'$: finite linear combination of the e_n

$$= \sum_{|n| \leq N} c_n e_n, \qquad c_n = \int_{\mathbb{R}/\mathbb{Z}} \psi'(x) e(-nx)\, dx$$

such that $\|\psi - \psi'\|_\infty \leq \varepsilon$.

$$\Downarrow$$

$$\left| c_0 - \int \psi \right| = \left| \int (\psi' - \psi) \right| \leq \varepsilon.$$

By $(\ast\ast)$, $\frac{1}{J} \sum_1^J \psi'(x_j) \to \int \psi'$.

$$\Rightarrow \exists\, J_0(\varepsilon) \text{ s.t. } \forall\, J \geq J_0(\varepsilon),$$

$$\left| \frac{1}{J} \sum_1^J \psi'(x_j) - \int \psi' \right| \leq \varepsilon.$$

Also, $\left| \frac{1}{J} \sum_1^J \psi'(x_j) - \frac{1}{J} \sum_1^J \psi(x_j) \right| \leq \varepsilon$

Thus, by the triangle inequality,

$$\left| \frac{1}{J} \sum_1^J \psi(x_j) - \int \psi \right| \leq 3\varepsilon.$$

Thus $(\ast)$ holds for ψ, so (x_j) equidistributes. $\square$

Corollary For $\alpha \in \mathbb{R} - \mathbb{Q}$, the fractional parts $[n\alpha] \in \mathbb{R}/\mathbb{Z}$, $(n \geq 1)$

$[x] :=$ image of $\underset{\mathbb{R}}{\overset{\cap}{x}}$ in $\mathbb{R}/\mathbb{Z}$

equidistribute: $\forall\, \varphi \in C_c(\mathbb{R}/\mathbb{Z})$

$$\frac{1}{N} \sum_{1}^{N} \varphi([n\alpha]) \longrightarrow \int_{\mathbb{R}/\mathbb{Z}} \varphi(x)\, dx.$$

In particular, $\{[n\alpha]\}$: dense in $\mathbb{R}/\mathbb{Z}$.

Proof By Weyl's criterion, we must check: $\forall\, n \in \mathbb{Z}_{\neq 0}$,

$$\Big| \frac{1}{N} \sum_{j=1}^{N} \underbrace{e_n(j\alpha)}_{= t^j,\ t := e(\underbrace{n\alpha}_{\notin \mathbb{Z} \text{ b/c } \alpha \notin \mathbb{Q}}) \neq 1} \Big| \longrightarrow 0 \quad \text{as } N \to \infty.$$

$\|$ geometric series evaluation

$$\left| \frac{1}{N} \frac{t^{N+1} - t}{t - 1} \right| \leq \frac{2}{N} \underbrace{\frac{1}{|t-1|}}_{< \infty} \longrightarrow 0 \quad \text{as } N \to \infty.$$

$\square$

More generally, Weyl's criterion holds for $X = L^2(G)$,
where G: compact topological abelian group,
μ = probability Haar measure on G
invariant under translation

Non-compact abelian groups

Examples (1) $G = \mathbb{R}$ additive group, Fourier transform

$$L^2(\mathbb{R}) \ni v = \int_{\xi \in \mathbb{R}} \hat{v}(\xi)\, e_\xi \, d\xi,$$

$$\hat{v}(\xi) = \int_{\mathbb{R}} v(x)\, e(-\xi x)\, dx$$

$$e_\xi(x) = e(\xi x)$$

Note · e_ξ are eigenfunctions of $\Delta = \frac{\partial^2}{\partial x^2}$

$$\Delta e_\xi = -(2\pi)^2 \xi^2 e_\xi$$

· e_ξ are eigenfunctions under translation operator $\rho(x)$ $(x \in \mathbb{R})$: $\rho(x) e_\xi = e(\xi x) e_\xi$

· $e_\xi \notin L^2(\mathbb{R})$ $\quad \int_{\mathbb{R}} |e_\xi(x)|^2 dx = \int_{\mathbb{R}} dx = \infty$

· the eigenfunctions e_ξ come in continuous families $(\xi \in \mathbb{R})$
rather than discrete families (like the e_n, $n \in \mathbb{Z}$)

· $\langle v_1, v_2 \rangle = \int_{\xi \in \mathbb{R}} \langle v_1, e_\xi \rangle \langle e_\xi, v_2 \rangle \, d\xi$
$(\forall v_1, v_2 \in H)$
Parseval

"Summary" $\quad L^2(\mathbb{R}/\mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} e_n, \quad L^2(\mathbb{R}) = \oint_{\xi \in \mathbb{R}} \mathbb{C} e_\xi \, d\xi$

means that Parseval holds.

(2) $G = \mathbb{R}^\times$ multiplicative group, Mellin transform

$$\begin{array}{c} \mathbb{R}^\times \\ \cong \\ \mathbb{R}^\times_+ \times \{\pm 1\}, \quad \mathbb{R}^\times_+ \xrightarrow[\log]{\cong} \mathbb{R} \end{array}$$

Spaces of lattices

<u>Definition</u> A <u>lattice</u> $\Lambda \leq \mathbb{R}^n$ is a subgroup s.t. $\exists$ basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ s.t. $\Lambda = \bigoplus_1^n \mathbb{Z} v_j$.

<u>Example</u> Standard lattice $\Lambda_0 := \mathbb{Z}^n \leq \mathbb{R}^n$, $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ (standard basis elements for $\mathbb{R}^n$) j

<u>Lemma</u> The following are equivalent for a subgroup $\Lambda \leq \mathbb{R}^n$:

(i) Λ is a lattice

(ii) Λ is discrete and cocompact

(iii) Λ is discrete, $\mathrm{vol}(\mathbb{R}^n/\Lambda) < \infty$

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as a space of row vectors

(row vector) $\cdot$ (matrix)

$$\{\text{lattices } \Lambda \leq \mathbb{R}^n\} \curvearrowleft GL_n(\mathbb{R})$$

$$g : \Lambda \mapsto \Lambda g = \{vg : v \in \Lambda\}$$

<u>Lemma</u> This action is transitive

<u>Proof</u> Let Λ: lattice, say $\Lambda = \bigoplus \mathbb{Z} v_j$. $(\Lambda_0 = \bigoplus \mathbb{Z} e_j)$

Choose $g \in GL_n(\mathbb{R})$ s.t. $e_j g = v_j$.

Then $\Lambda_0 g = \Lambda$. $\square$

The action is not simple, i.e., g is not unique:

$$\mathrm{Stab}_{GL_n(\mathbb{R})}(\Lambda_0) = \{g : \Lambda_0 g = \Lambda_0\} = GL_n(\mathbb{Z}). \quad \square$$

$$\Updownarrow$$

$$\Lambda_0 = \Lambda_0 g^{-1}$$

We may thus identify

$$X_n := \{\text{lattices in } \mathbb{R}^n\} \longleftrightarrow GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R})$$
$$\Lambda_0 g \longleftrightarrow GL_n(\mathbb{Z})\, g$$

<u>Defn</u> We call $\Lambda \in X_n$ <u>unimodular</u> if $\mathrm{vol}(\mathbb{R}^n/\Lambda) = 1$.
Equivalently, $\Lambda = \Lambda_0 g$ for some $g \in SL_n(\mathbb{R})$.

Then
$$X_n^{(1)} := \left\{ \begin{matrix}\text{unimodular}\\ \text{lattices in } \mathbb{R}^n\end{matrix} \right\} \longleftrightarrow SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})$$

$$\begin{array}{ccc} X_n^{(1)} & \longrightarrow & X_n \\ & & \downarrow\ \Lambda \mapsto \mathrm{vol}(\mathbb{R}^n/\Lambda) \\ & & \mathbb{R}_+^\times \end{array}$$

In fact, $X_n \cong X_n^{(1)} \times \mathbb{R}_+^\times$
$$\Lambda \mapsto (c\Lambda, \mathrm{vol}), \qquad c > 0 \text{ s.t. } c\Lambda: \text{unimodular}.$$

Thus many questions involving X_n can be reduced to questions involving $X_n^{(1)}$ and $\mathbb{R}_+^\times$.

<u>Main goal</u> "construct nice bases" for spaces of functions on X_n, $X_n^{(1)}$.

<u>Fact</u> $X_n^{(1)}$ admits a (unique) $SL_n(\mathbb{R})$-invariant probability measure μ
X_n admits a nonzero $GL_n(\mathbb{R})$-invariant measure

$$\rightsquigarrow \quad L^2(X_n), \qquad L^2(X_n^{(1)})$$
$$\not\ni 1 \qquad\qquad \ni 1$$

"nice": characterized by analogy to $L^2(\mathbb{R}/\mathbb{Z})$

- eigenfunctions of certain differential operators
- irreducibility under $G = SL_n(\mathbb{R})$ or $GL_n(\mathbb{R})$, acting via right translation

"basis": mixture of what happened for $L^2(\mathbb{R}/\mathbb{Z})$, $L^2(\mathbb{R})$

Why care? Many problems in number theory, Diophantine analysis, (...), boil down to equidistribution problems involving X_n, $X_n^{(1)}$!

Example

Littlewood conjecture $\forall\, \alpha, \beta \in \mathbb{R}$

$$\liminf_{n \to \infty} n \cdot \|n\alpha\| \cdot \|n\beta\| = 0$$

$$\left(\|x\| := \min_{\ell \in \mathbb{Z}} |x - \ell|\right)$$

$$\Uparrow$$

Conjecture Let $A = \left\{ \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} : \begin{matrix} x_i > 0 \\ x_1 x_2 x_3 \end{matrix} \right\} \leq SL_3(\mathbb{R})$

Let $z \in X_3^{(1)}$ s.t. zA is precompact.
Then zA is closed.

idea $n\|n\alpha\| \cdot \|n\beta\| =$ smallest number > 0 of the form $\pm \underbrace{n(n\alpha + m)(n\beta + \ell)}_{\substack{\| \\ \text{value taken by} \\ (x,y,z) \mapsto xyz}}$

on the lattice $\Lambda_{\alpha\beta} = \langle (1,0,0), (\alpha,1,0), (\beta,0,1) \rangle$

How to construct functions on X_n OR $X_n^{(1)}$?

Eisenstein series give a rich class of examples of such constructions.

Ex Let $f \in C_c(\mathbb{R}^n - \{0\})$.

Define $\mathrm{Eis}[f] : X_n \to \mathbb{C}$

$$\Lambda \mapsto \sum_{v \in \Lambda - \{0\}} f(v)$$

"Siegel Eisenstein series"

Ex Same definition, but restrict to primitive vectors $v \in \Lambda$, i.e., those of the form

$v = v_1$ for some basis $v_1, \ldots, v_n$ of Λ.

Ex Write $n = n_1 + n_2$

$$X_{n_1,n_2} := \left\{ \begin{array}{l} \text{triples } (V_1, \Lambda_1, \Lambda_2): \\ \quad V_1 : n_1\text{-dim'l subspace of } \mathbb{R}^n \\ \quad \Lambda_1 : \text{lattice in } V_1 \\ \quad \Lambda_2 : \text{lattice in } V/V_1 \end{array} \right\} \circlearrowleft GL_n(\mathbb{R})$$

Given $f \in C_c(X_{n_1,n_2})$,

$\mathrm{Eis}[f] : X_n \to \mathbb{C}$

$$\Lambda \mapsto \sum_{\substack{\text{pairs } (\Lambda_1, \Lambda_2 \bmod \Lambda_1) \\ \text{of subgroups } \Lambda_1, \Lambda_2 \leq \Lambda \text{ s.t. } \Lambda = \Lambda_1 \oplus \Lambda_2 \\ \dim_{\mathbb{Z}}(\Lambda_j) = n_j}} f(V_1, \Lambda_1, \Lambda_2 \bmod V_1) \qquad V_1 := \mathbb{R}\text{-span of } \Lambda_1$$

Work of Langlands and other (recent Abel prize):

$$L^2(X_n^{(1)}) = \left(\begin{array}{l} \text{Eisenstein series as} \\ \text{above attached} \\ \text{to functions on } X_{n_1,n_2} \end{array} \right) \oplus \left(\begin{array}{c} \text{orthogonal} \\ \text{complement} \end{array} \right)$$

well understood

interesting subspace